Pensieve header: Analysis of k=2 invariants in QU: Brute determination of the relations between the k=2 coefficients.

In[ ]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
<< KnotTheory`
<< "SL2PortfolioProgram.m"
```

ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

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ToFileName: String or list of strings expected at position 1 in ToFileName[{File, WikiLink, mathematica}].

ToFileName: String or list of strings expected at position 1 in ToFileName[{File, QuantumGroups}].

```
Loading KnotTheory` version of January 20, 2015, 10:42:19.1122.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[ ]:=
```
OverbayP2Data = Get["C:\\drorbn\\AcademicPensieve\\People\\Overbay\\OverbayP2Data.m"];
OP2[K_Knot] := K /. OverbayP2Data /. T → T^(1/2);
```

In[ ]:=
```
{Length[OverbayP2Data], Last[OverbayP2Data]}
```

Out[ ]=
$$\left\{35,\ \text{Knot}[8,21] \to 2670 + \frac{1}{T^{14}} + \frac{4}{T^{12}} - \frac{60}{T^{10}} + \frac{276}{T^8} - \frac{775}{T^6} + \frac{1550}{T^4} - \frac{2331}{T^2} - 2331\,T^2 + 1550\,T^4 - 775\,T^6 + 276\,T^8 - 60\,T^{10} + 4\,T^{12} + T^{14}\right\}$$

In[ ]:=
```
$p = 5; $k = 2; $U = QU;
```

In[ ]:=
```
SCΘ[p_] := Collect[ CΘ@@𝒪CU[{y, a, x}] , p] /. {CU → Times, γ | ħ → 1}, ϵ, Simplify];
SQΘ[p_] := Collect[ QΘ@@𝒪QU[{y, a, x}] , p] /. {QU → Times, γ | ħ → 1}, ϵ, Simplify];
```

In[ ]:=
```
𝔼[L_, Q_, P_]$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
𝔼d_→r_[L_, Q_, P_]$k_ := 𝔼d→r@@𝔼[L, Q, P]$k;
𝔼3@𝔼[ω_, L_, Q_, Ps_] := CF /@ 𝔼[L, ω^-1 Q, ω^-1 (ω^-4 ϵ)^(-1+Range@Length@Ps .Ps)]$k;
𝔼4@𝔼[L_, Q_, P_] := Module[
   {ω = Normal[P]^-1 /. ϵ → 0, Ps = CoefficientList[P, ϵ]},
   CF /@ 𝔼[ω, L, ω Q, ω^(-3+4 Range@Length@Ps) Ps]];
𝔼3@𝔼sp___[as___] := 𝔼3@𝔼[as] /. 𝔼 → 𝔼sp;
𝔼4@𝔼sp___[as___] := 𝔼4@𝔼[as] /. 𝔼 → 𝔼sp;
```

In[ ]:=
```
Clear[QP, ω];
QP[Knot[n_, k_]] := QP[Knot[n, k]] = Collect[Module[{fname},
      fname = "../SL2Invariant/k=2/Data/" <> ToString[n] <> "_" <> ToString[k] <> ".m";
      Collect[𝔼3[Get[fname]][[2, 2]]][[3]] // Normal, ϵ, Simplify]
    ], ϵ, CF];
ω[K_Knot] := ω[K] = Factor[(QP@K /. ϵ → 0)^-1];
c_{k_,d_}[K_Knot] :=
 Factor[SeriesCoefficient[QP[K], {y, 0, 0}, {ϵ, 0, k}, {a, 0, d}] ω[K]^(1+2k-d)]
```

In[ ]:=
```
H[p_] := If[TrueQ@Simplify[p == (p /. T → 1/T)],
   σ@@CoefficientList[Expand@Together[p] /. T^n_ /; n < 0 :↦ 0, T], p];
H[p_] := If[TrueQ@Simplify[p == (p /. T → 1/T)],
   Style[Expand@Together[p] /. T^n_ /; n < 0 :↦ 0, Background → Yellow], p];
H[p_] := If[TrueQ@Simplify[p == (p /. T → 1/T)], Style[p, Background → Yellow], p];
```

In[ ]:=
```
MatrixForm[Table[
  H /@ Factor /@ {ω[K], c_{0,0}[K], c_{1,0}[K], c_{1,1}[K], c_{2,0}[K], c_{2,1}[K], c_{2,2}[K] OP2[K]},
  {K, AllKnots[{3, 7}]}
 ]]
```

Out[ ]//MatrixForm=

$\frac{1-T+T^2}{T}$	1	$\frac{(-1+T)(2-T+T^2)}{T^2}$	$\frac{2(-1+T)(1+T)}{T}$	
$-\frac{1-3T+T^2}{T}$	1	$\frac{(-1+T)(1+T)(1-3T+T^2)}{T^2}$	$-\frac{2(-1+T)(1+T)}{T}$	
$\frac{1-T+T^2-T^3+T^4}{T^2}$	1	$\frac{(-1+T)(4-3T+5T^2-3T^3+3T^4-T^5+T^6)}{T^4}$	$\frac{2(-1+T)(1+T)(2-T+2T^2)}{T^2}$	
$\frac{2-3T+2T^2}{T}$	1	$-\frac{(-1+T)(-9+11T-7T^2+T^3)}{T^2}$	$\frac{4(-1+T)(1+T)}{T}$	
$-\frac{(-2+T)(-1+2T)}{T}$	1	$\frac{(-1+T)(5-11T-T^2+3T^3)}{T^2}$	$-\frac{4(-1+T)(1+T)}{T}$	
$-\frac{1-3T+3T^2-3T^3+T^4}{T^2}$	1	$\frac{(-1+T)(3-12T+16T^2-12T^3+4T^4-2T^6+T^7)}{T^4}$	$-\frac{2(-1+T)(1+T)(2-3T+2T^2)}{T^2}$	
$\frac{1-3T+5T^2-3T^3+T^4}{T^2}$	1	$\frac{(-1+T)(1+T)(2-3T+2T^2)(1-3T+5T^2-3T^3+T^4)}{T^4}$	$\frac{2(-1+T)(1+T)(2-3T+2T^2)}{T^2}$	
$\frac{1-T+T^2-T^3+T^4-T^5+T^6}{T^3}$	1	$\frac{(-1+T)(6-5T+9T^2-7T^3+9T^4-6T^5+6T^6-3T^7+3T^8-T^9+T^{10})}{T^6}$	$\frac{2(-1+T)(1+T)(3-2T+4T^2-2T^3+3T^4)}{T^3}$	$\frac{(-1+T)(-36}{}$
$\frac{3-5T+3T^2}{T}$	1	$-\frac{(-1+T)(-23+36T-24T^2+5T^3)}{T^2}$	$\frac{6(-1+T)(1+T)}{T}$	
$\frac{2-3T+3T^2-3T^3+2T^4}{T^2}$	1	$\frac{(-1+T)(-1+7T-13T^2+24T^3-32T^4+35T^5-27T^6+17T^7)}{T^4}$	$\frac{2(-1+T)(1+T)(4-3T+4T^2)}{T^2}$	$($
$\frac{4-7T+4T^2}{T}$	1	$\frac{4(-1+T)(-2+11T-17T^2+10T^3)}{T^2}$	$\frac{8(-1+T)(1+T)}{T}$	
$\frac{2-4T+5T^2-4T^3+2T^4}{T^2}$	1	$-\frac{(-1+T)(-17+41T-65T^2+65T^3-49T^4+25T^5-9T^6+T^7)}{T^4}$	$\frac{8(-1+T)(1+T)(1-T+T^2)}{T^2}$	$(-1+$
$-\frac{1-5T+7T^2-5T^3+T^4}{T^2}$	1	$\frac{(-1+T)(3-22T+53T^2-53T^3+25T^4-T^5-4T^6+T^7)}{T^4}$	$-\frac{2(-2+T)(-1+T)(1+T)(-1+2T)}{T^2}$	
$\frac{1-5T+9T^2-5T^3+T^4}{T^2}$	1	$\frac{(-1+T)(2-13T+27T^2-9T^3-31T^4+33T^5-13T^6+2T^7)}{T^4}$	$\frac{2(-2+T)(-1+T)(1+T)(-1+2T)}{T^2}$	$(-$

In[ ]:=
```
p1[K_Knot] := p1[K] = Factor[(T (-c_{1,0}[K] + ω[K] T ∂_T ω[K]))/(T - 1)^2];
```

In[•]:=
```
MyCollect[ℰ_, vs_List] := MyCollect[ℰ, vs, Identity];
MyCollect[ℰ_, vs_List, simp_] :=
 Total[CoefficientRules[ℰ, vs] /. ((ps_ → c_) :↦ simp[c] Times @@ (vs^ps))]
```

In[•]:=
```
Monitor[Total[Table[
   Simplify[
    (c0,0[K] == 1) ∧ (2 T ∂T ω[K] == c1,1[K]) ∧ (0 == c2,2[K] (-3 c1,0[K] ∂T ω[K] + ω[K] ∂T c1,0[K]) +
        c2,1[K] (-ω[K] ∂T ω[K] + 2 T (∂T ω[K])^2 - T ω[K] ∂T,T ω[K])) ∧ (c2,1[K] == ω[K] c2,2[K] +
        2 (-1 + T)^2 ω[K] ∂T p1[K] - (2 (-1 + T) p1[K] (-(1 + T) ω[K] + 3 (-1 + T) T ∂T ω[K]))/T),
   {K, AllKnots[{3, 10}]}]],
 K]
```

Out[•]= 249 True

In[•]:=
```
MatrixForm[Table[
  H /@ Factor /@
    {ω[K], p1[K], -2 c2,0[K] + ω[K] c2,1[K], (c2,1[K] - ω[K] c2,2[K])/(-2 T + 2/T), c2,2[K]/2, OP2[K]},
  {K, AllKnots[{3, 7}]}
 ]]
```

Out[•]//MatrixForm=

$\frac{1-T+T^2}{T}$	$\frac{1+T^2}{T}$	$3-12\,T+26\,T$
$-\frac{1-3\,T+T^2}{T}$	0	$(1-3\,T+T^2)$
$\frac{1-T+T^2-T^3+T^4}{T^2}$	$\frac{(1+T^2)\,(2+T^2+2\,T^4)}{T^3}$	$5-20\,T+55\,T^2-120\,T^3+217\,T^4-338\,T^5+450\,T^6-$
$\frac{2-3\,T+2\,T^2}{T}$	$\frac{5-4\,T+5\,T^2}{T}$	$-\frac{10-120\,T+487\,T^2-1054\,T}{}$
$-\frac{(-2+T)\,(-1+2\,T)}{T}$	$\frac{1-4\,T+T^2}{T}$	$14-16\,T-293\,T^2+1098\,T^3$
$-\frac{1-3\,T+3\,T^2-3\,T^3+T^4}{T^2}$	$\frac{1-4\,T+4\,T^2-4\,T^3+4\,T^4-4\,T^5+T^6}{T^3}$	$3-21\,T+49\,T^2+15\,T^3-433\,T^4+1543\,T^5-3431\,T^6+5482\,T^7-64$
$\frac{1-3\,T+5\,T^2-3\,T^3+T^4}{T^2}$	0	$(1-3\,T+5\,T^2-3\,T^3+T^4)\,(4-21\,T+38\,T^2+28\,T^3-284$
$\frac{1-T+T^2-T^3+T^4-T^5+T^6}{T^3}$	$\frac{(1+T^2)\,(3+2\,T^2+4\,T^4+2\,T^6+3\,T^8)}{T^5}$	$7-28\,T+77\,T^2-168\,T^3+322\,T^4-560\,T^5+891\,T^6-1310\,T^7+1777\,T^8-2238\,T^9+2604\,T^{10}-277$
$\frac{3-5\,T+3\,T^2}{T}$	$\frac{2\,(7-8\,T+7\,T^2)}{T}$	$-\frac{129-1177\,T+4421\,T^2-9226\,T^3}{}$
$\frac{2-3\,T+3\,T^2-3\,T^3+2\,T^4}{T^2}$	$-\frac{9-8\,T+16\,T^2-12\,T^3+16\,T^4-8\,T^5+9\,T^6}{T^3}$	$-\frac{18-208\,T+917\,T^2-2666\,T^3+6049\,T^4-11\,283\,T^5+17\,671\,T^6-23\,356\,T^7+25}{}$
$\frac{4-7\,T+4\,T^2}{T}$	$-\frac{8\,(3-4\,T+3\,T^2)}{T}$	$-\frac{2\,(176-1808\,T+7189\,T^2-15\,350\,T}{}$
$\frac{2-4\,T+5\,T^2-4\,T^3+2\,T^4}{T^2}$	$\frac{9-16\,T+29\,T^2-28\,T^3+29\,T^4-16\,T^5+9\,T^6}{T^3}$	$-\frac{18-264\,T+1548\,T^2-5680\,T^3+15\,107\,T^4-31\,152\,T^5+51\,476\,T^6-69\,252\,T^7+76}{}$
$-\frac{1-5\,T+7\,T^2-5\,T^3+T^4}{T^2}$	$\frac{1-8\,T+19\,T^2-20\,T^3+19\,T^4-8\,T^5+T^6}{T^3}$	$3-35\,T+128\,T^2+105\,T^3-2610\,T^4+11\,225\,T^5-28\,031\,T^6+47\,186\,T^7-55$
$\frac{1-5\,T+9\,T^2-5\,T^3+T^4}{T^2}$	$-\frac{3-8\,T+3\,T^2}{T}$	$4-55\,T+310\,T^2-805\,T^3+86\,T^4+6349\,T^5-22\,686\,T^6+43\,610\,T^7-53$

```
In[•]:= MatrixForm[mat = Table[
          {q1 = ω[K], q2 = p1[K], (* T∂_T ω[K] / (T - T^-1), T∂_T(T∂_T ω[K]), *) q3 = -2 c_{2,0}[K] + ω[K] c_{2,1}[K],
           q4 = ω[K] (c_{2,1}[K] - ω[K] c_{2,2}[K]) / (-2 T + 2/T), q5 = ω[K]^2 c_{2,2}[K] / 2, q6 = ω[K]^2 p1[K]};
          Times @@@ Subsets[Together /@ {q1, q2, q3, q4, q5, q6}, 5] /. T → -1,
          {K, AllKnots[{3, 10}]}
         ]];
       Dimensions[mat]
       MatrixRank[mat]
```

```
Out[•]= {249, 63}
```

```
Out[•]= 63
```

```
In[•]:= MatrixForm[mat = Table[
          {q1, q2, q3, q4, q5} = Together /@ {ω[K], p1[K], (* T∂_T ω[K] / (T - T^-1), T∂_T(T∂_T ω[K]), *)
             -2 c_{2,0}[K] + ω[K] c_{2,1}[K], ω[K] (c_{2,1}[K] - ω[K] c_{2,2}[K]) / (-2 T + 2/T), ω[K]^2 c_{2,2}[K] / 2};
          Join @@ Table[q1^k {q2, q3, q4, q5}, {k, 0, 10}] /. T → -1,
          {K, AllKnots[{3, 10}]}
         ]];
       Dimensions[mat]
       MatrixRank[mat]
```

```
Out[•]= {249, 44}
```

```
Out[•]= 44
```

```
In[•]:= Most /@ (Join @@ (Permutations /@ (PadRight[#, 6] & /@ IntegerPartitions[3])))
```

```
Out[•]= {{3, 0, 0, 0, 0}, {0, 3, 0, 0, 0}, {0, 0, 3, 0, 0}, {0, 0, 0, 3, 0},
         {0, 0, 0, 0, 3}, {0, 0, 0, 0, 0}, {2, 1, 0, 0, 0}, {2, 0, 1, 0, 0},
         {2, 0, 0, 1, 0}, {2, 0, 0, 0, 1}, {2, 0, 0, 0, 0}, {1, 2, 0, 0, 0},
         {1, 0, 2, 0, 0}, {1, 0, 0, 2, 0}, {1, 0, 0, 0, 2}, {1, 0, 0, 0, 0},
         {0, 2, 1, 0, 0}, {0, 2, 0, 1, 0}, {0, 2, 0, 0, 1}, {0, 2, 0, 0, 0}, {0, 1, 2, 0, 0},
         {0, 1, 0, 2, 0}, {0, 1, 0, 0, 2}, {0, 1, 0, 0, 0}, {0, 0, 2, 1, 0}, {0, 0, 2, 0, 1},
         {0, 0, 2, 0, 0}, {0, 0, 1, 2, 0}, {0, 0, 1, 0, 2}, {0, 0, 1, 0, 0}, {0, 0, 0, 2, 1},
         {0, 0, 0, 2, 0}, {0, 0, 0, 1, 2}, {0, 0, 0, 1, 0}, {0, 0, 0, 0, 2}, {0, 0, 0, 0, 1},
         {1, 1, 1, 0, 0}, {1, 1, 0, 1, 0}, {1, 1, 0, 0, 1}, {1, 1, 0, 0, 0}, {1, 0, 1, 1, 0},
         {1, 0, 1, 0, 1}, {1, 0, 1, 0, 0}, {1, 0, 0, 1, 1}, {1, 0, 0, 1, 0}, {1, 0, 0, 0, 1},
         {0, 1, 1, 1, 0}, {0, 1, 1, 0, 1}, {0, 1, 1, 0, 0}, {0, 1, 0, 1, 1}, {0, 1, 0, 1, 0},
         {0, 1, 0, 0, 1}, {0, 0, 1, 1, 1}, {0, 0, 1, 1, 0}, {0, 0, 1, 0, 1}, {0, 0, 0, 1, 1}}
```

```
In[•]:= MatrixForm[mat = Table[
          vs = (Together /@
              {ω[K], p1[K], T ∂T p1[K] / (T - T^-1), T ∂T ω[K] / (T - T^-1), T ∂T (T ∂T ω[K]), -2 c2,0[K] + ω[K] c2,1[K],
               ω[K] (c2,1[K] - ω[K] c2,2[K]) / (-2 T + 2/T), ω[K]^2 c2,2[K] / 2 (*,OP2[K]*)}) /. T → (μ = 10^9);
          PS = Select[Most /@ (Join @@ (Permutations /@ (PadRight[#, 1 + Length[vs]] & /@
                  IntegerPartitions[4]))), (Total[#[[-3 ;; -1]]] == 1 ∧ #[[3]] + #[[5]] ≤ 1) &];
          Table[Times @@ (v /@ Range@Length@vs)^ps, {ps, PS}] /. v[i_] :> vs[[i]],
          {K, AllKnots[{3, 10}]}
         ]];
        Dimensions[mat]
        MatrixRank[mat]

Out[•]= {249, 120}

Out[•]= 119

In[•]:= NullSpace[mat].Table[Times @@ (v /@ Range@Length@vs)^ps, {ps, PS}]

Out[•]= {-(1 000 000 002 000 000 001 v[1] v[4]^2 v[7]) / 1 500 000 000 + (1 000 000 000 v[1]^2 v[5] v[7]) / 2 999 999 994 000 000 003 -
  (1 000 000 000 v[1] v[2] v[8]) / 2 999 999 994 000 000 003 - 1/3 v[1] v[3] v[8] + v[2] v[4] v[8]}

In[•]:= 3 (μ - 1)^2 == 2 999 999 994 000 000 003

Out[•]= True

In[•]:= Monitor[Total[Table[
          Simplify[(c0,0[K] == 1) ∧ (2 T ∂T ω[K] == c1,1[K]) ∧
            (0 == -((T + 1)^2 / (3 T/2)) ω[K] (T ∂T ω[K] / (T - T^-1))^2 (ω[K] (c2,1[K] - ω[K] c2,2[K]) / (-2 T + 2/T)) +
              T ω[K]^2 (T ∂T (T ∂T ω[K])) (ω[K] (c2,1[K] - ω[K] c2,2[K]) / (-2 T + 2/T)) / (3 (T - 1)^2) -
              T ω[K] p1[K] (ω[K]^2 c2,2[K] / 2) / (3 (T - 1)^2) -
              1/3 ω[K] (T ∂T p1[K] / (T - T^-1)) (ω[K]^2 c2,2[K] / 2) + p1[K] (T ∂T ω[K] / (T - T^-1)) (ω[K]^2 c2,2[K] / 2)),
          {K, AllKnots[{3, 10}]}]],
        K]

Out[•]= 249 True
```

```
In[•]:= rel1 = Simplify@ReleaseHold[Hold[-((T+1)^2/(3 T/2)) ω[K] ((T ∂_T ω[K])/(T-T^-1))^2 (ω[K] (c_{2,1}[K]-ω[K] c_{2,2}[K])/(-2 T+2/T)) +
    (T ω[K]^2 (T ∂_T (T ∂_T ω[K])) (ω[K] (c_{2,1}[K]-ω[K] c_{2,2}[K])/(-2 T+2/T)))/(3 (T-1)^2) - (T ω[K] p1[K] (ω[K]^2 c_{2,2}[K]/2))/(3 (T-1)^2) -
    1/3 ω[K] ((T ∂_T p1[K])/(T-T^-1)) (ω[K]^2 c_{2,2}[K]/2) + p1[K] ((T ∂_T ω[K])/(T-T^-1)) (ω[K]^2 c_{2,2}[K]/2)] /. K → T]

Out[•]= 1/(6 (-1+T)^3 (1+T)) T ω[T]^2 (-(-1+T) p1[T] c_{2,2}[T] ((1+T) ω[T] - 3 (-1+T) T ω'[T]) +
    T (2 T^2 c_{2,1}[T] ω'[T]^2 + T ω[T]^2 c_{2,2}[T] (ω'[T] + T ω''[T]) -
      ω[T] (c_{2,2}[T] ((-1+T)^2 p1'[T] + 2 T^2 ω'[T]^2) + T c_{2,1}[T] (ω'[T] + T ω''[T]))))

In[•]:= MyCollect[rel1, {ω[T], ω'[T], ω''[T], p1[T], p1'[T]}, Simplify]

Out[•]= -(T p1[T] ω[T]^3 c_{2,2}[T])/(6 (-1+T)^2) + (T^2 ω[T]^3 c_{2,2}[T] p1'[T])/(6-6 T^2) - (T^3 ω[T]^3 c_{2,1}[T] ω'[T])/(6 (-1+T)^3 (1+T)) +
    (T^2 p1[T] ω[T]^2 c_{2,2}[T] ω'[T])/(2 (-1+T^2)) + (T^3 ω[T]^4 c_{2,2}[T] ω'[T])/(6 (-1+T)^3 (1+T)) + (T^4 ω[T]^2 c_{2,1}[T] ω'[T]^2)/(3 (-1+T)^3 (1+T)) -
    (T^4 ω[T]^3 c_{2,2}[T] ω'[T]^2)/(3 (-1+T)^3 (1+T)) - (T^4 ω[T]^3 c_{2,1}[T] ω''[T])/(6 (-1+T)^3 (1+T)) + (T^4 ω[T]^4 c_{2,2}[T] ω''[T])/(6 (-1+T)^3 (1+T))

In[•]:= rel2 = Simplify[
    rel1 /. {p1[T] -> (T (-c_{1,0}[T] + ω[T] T ∂_T ω[T]))/(T-1)^2, p1'[T] → D[(T (-c_{1,0}[T] + ω[T] T ∂_T ω[T]))/(T-1)^2, T]}]

Out[•]= -1/(6 (-1+T)^3 (1+T)) T^3 ω[T]^2 (ω'[T] (3 c_{1,0}[T] c_{2,2}[T] - 2 T c_{2,1}[T] ω'[T]) +
    ω[T] (-c_{2,2}[T] c_{1,0}'[T] + c_{2,1}[T] (ω'[T] + T ω''[T]))))

In[•]:= MyCollect[6 (-1+T)^3/(T^3 ω[T]^2) (1+T) rel2, {ω[T], ω'[T], ω''[T]}, Simplify]

Out[•]= -ω[T] c_{2,1}[T] ω'[T] - 3 c_{1,0}[T] c_{2,2}[T] ω'[T] +
    2 T c_{2,1}[T] ω'[T]^2 + ω[T] c_{2,2}[T] c_{1,0}'[T] - T ω[T] c_{2,1}[T] ω''[T]

In[•]:= MyCollect[6 (-1+T)^3/(T^3 ω[T]^2) (1+T) rel2, {c_{2,1}[T], c_{2,2}[T]}, Factor]

Out[•]= c_{2,2}[T] (-3 c_{1,0}[T] ω'[T] + ω[T] c_{1,0}'[T]) + c_{2,1}[T] (-ω[T] ω'[T] + 2 T ω'[T]^2 - T ω[T] ω''[T])
```

```
In[•]:= MatrixForm[mat = Table[
          vs = (Together /@
              {ω[K], p1[K], T ∂T p1[K] / (T - T^-1), T ∂T ω[K] / (T - T^-1), T ∂T (T ∂T ω[K]), -2 c2,0[K] + ω[K] c2,1[K],
               ω[K] (c2,1[K] - ω[K] c2,2[K]) / (-2 T + 2/T), ω[K]^2 c2,2[K] / 2 (*,OP2[K]*)}) /. T → (μ = 10^9);
          PS = Select[Most /@ (Join @@ (Permutations /@ (PadRight[#, 1 + Length[vs]] & /@
                  IntegerPartitions[5]))), (Total[#[[-3 ;; -1]]] == 1 ∧ #[[3]] + #[[5]] ≤ 1) &];
          Table[Times @@ (v /@ Range@Length@vs)^ps, {ps, PS}] /. v[i_] :> vs[[i]],
          {K, AllKnots[{3, 10}]}
         ]];
        Dimensions[mat]
        MatrixRank[mat]

Out[•]= {249, 225}

Out[•]= 221

In[•]:= (ns5 = NullSpace[mat].Table[Times @@ (v /@ Range@Length@vs)^ps, {ps, PS}]) // Column

Out[•]= 2 000 000 004 000 000 002 v[1]^2 v[4]^2 v[7] / 2 999 999 994 000 000 003 + 1 000 000 002 000 000 001 v[1] v[4]^3 v[7] / 500 000 000 -
 1 000 000 000 000 000 000 v[1]^3 v[5] v[7] / 2 999 999 988 000 000 017 999 999 988 000 000 003 - 1 000 000 000 v[1]^2 v[4] v[5] v[7] / 999 999 998 000 000 001 + 1 000 000 000 000 000 000 v[1]^2 v[2] v[8] / 2 999 999 988 000 000 017 999 999 988 000 000 003 +
 1 000 000 000 v[1]^2 v[3] v[8] / 2 999 999 994 000 000 003 + v[1] v[3] v[4] v[8] - 3 v[2] v[4]^2 v[8]
- 1 000 000 002 000 000 001 v[1]^2 v[4]^2 v[7] / 1 500 000 000 + 1 000 000 000 v[1]^3 v[5] v[7] / 2 999 999 994 000 000 003 -
 1 000 000 000 v[1]^2 v[2] v[8] / 2 999 999 994 000 000 003 - 1/3 v[1]^2 v[3] v[8] + v[1] v[2] v[4] v[8]
 1 000 000 002 000 000 001 v[1] v[2] v[4]^2 v[7] / 500 000 000 - 1 000 000 000 v[1]^2 v[2] v[5] v[7] / 999 999 998 000 000 001 +
 1 000 000 000 v[1] v[2]^2 v[8] / 999 999 998 000 000 001 + v[1] v[2] v[3] v[8] - 3 v[2]^2 v[4] v[8]
- 1 000 000 002 000 000 001 v[1] v[4]^2 v[7] / 1 500 000 000 + 1 000 000 000 v[1]^2 v[5] v[7] / 2 999 999 994 000 000 003 -
 1 000 000 000 v[1] v[2] v[8] / 2 999 999 994 000 000 003 - 1/3 v[1] v[3] v[8] + v[2] v[4] v[8]

In[•]:= Together[ns5 / (-1 000 000 002 000 000 001 v[1] v[4]^2 v[7] / 1 500 000 000 + 1 000 000 000 v[1]^2 v[5] v[7] / 2 999 999 994 000 000 003 -
        1 000 000 000 v[1] v[2] v[8] / 2 999 999 994 000 000 003 - 1/3 v[1] v[3] v[8] + v[2] v[4] v[8])]

Out[•]= {(-1 000 000 000 v[1] - 2 999 999 994 000 000 003 v[4]) / 999 999 998 000 000 001, v[1], -3 v[2], 1}
```

```
In[•]:= MatrixForm[mat = Table[
          vs = (Together /@ {ω[K], p1[K], T ∂T p1[K] / (T - T^-1), T ∂T ω[K] / (T - T^-1),
                T ∂T (T ∂T ω[K]), T ∂T (T ∂T (T ∂T ω[K])), -2 c2,0[K] + ω[K] c2,1[K],
                ω[K] (c2,1[K] - ω[K] c2,2[K]) / (-2 T + 2/T) (*,ω[K]^2 c2,2[K]/2,OP2[K]*)}) /. T → (μ = 10^4);
          PS = Select[Most /@ (Join @@ (Permutations /@
                 (PadRight[#, 1 + Length[vs]] & /@ IntegerPartitions[5]))),
            (Total[#[[-2 ;; -1]]] == 1 ∧ #[[3]] + #[[5]] ≤ 1 ∧ #[[6]] ≤ 1) &];
          Table[Times @@ (v /@ Range@Length@vs)^ps, {ps, PS}] /. v[i_] :> vs[[i]],
          {K, AllKnots[{3, 10}]}
        ]];
      Dimensions[mat]
      MatrixRank[mat]

Out[•]= {249, 230}

Out[•]= 217

In[•]:= MatrixForm[mat = Table[
          vs = (Together /@ {ω[K], p1[K], T ∂T p1[K] / (T - T^-1), T ∂T ω[K] / (T - T^-1),
                T ∂T (T ∂T ω[K]), T ∂T (T ∂T (T ∂T ω[K])), -2 c2,0[K] + ω[K] c2,1[K],
                ω[K] (c2,1[K] - ω[K] c2,2[K]) / (-2 T + 2/T) (*,ω[K]^2 c2,2[K]/2,OP2[K]*)}) /. T → (μ = 10^9);
          PS = Select[Most /@ (Join @@ (Permutations /@
                 (PadRight[#, 1 + Length[vs]] & /@ IntegerPartitions[3]))),
            (Total[#[[-2 ;; -1]]] == 1 ∧ #[[3]] + #[[5]] ≤ 2 ∧ #[[6]] ≤ 1) &];
          Table[Times @@ (v /@ Range@Length@vs)^ps, {ps, PS}] /. v[i_] :> vs[[i]],
          {K, AllKnots[{3, 10}]}
        ]];
      Dimensions[mat]
      MatrixRank[mat]

Out[•]= {249, 54}

Out[•]= 54
```

```
In[•]:= MatrixForm[mat = Table[
          vs = (Together /@ {ω[K], p1[K], T ∂T p1[K] / (T - T^-1), T ∂T ω[K] / (T - T^-1),
                T ∂T (T ∂T ω[K]), T ∂T (T ∂T (T ∂T ω[K])), -2 c2,0[K] + ω[K] c2,1[K],
                ω[K] (c2,1[K] - ω[K] c2,2[K]) / (-2 T + 2/T) (*,ω[K]^2 c2,2[K]/2,OP2[K]*)}) /. T → (μ = 10^9);
          PS = Select[Most /@ (Join @@ (Permutations /@
                 (PadRight[#, 1 + Length[vs]] & /@ IntegerPartitions[3]))),
            (Total[#[[-2 ;; -1]]] ≤ 1 ∧ #[[3]] + #[[5]] ≤ 2 ∧ #[[6]] ≤ 1) &];
          Table[Times @@ (v /@ Range@Length@vs)^ps, {ps, PS}] /. v[i_] :> vs[[i]],
          {K, AllKnots[{3, 10}]}
        ]];
      Dimensions[mat]
      MatrixRank[mat]

Out[•]= {249, 127}

Out[•]= 123

In[•]:= NullSpace[mat].Table[Times @@ (v /@ Range@Length@vs)^ps, {ps, PS}] // Column

Out[•]= ... 1 ...
        ... 1 ...
        ... 1 ...
        -1 000 000 000 v[1]^2 v[2] / 2 999 999 994 000 000 003 - 1/3 v[1]^2 v[3] + v[1] v[2] v[4] - 1 000 000 000 v[8] / 2 999 999 994 000 000 003

        large output | show less | show more | show all | set size limit...

In[•]:= 3 × 999 999 998 000 000 001 / 1 000 000 000
         (-1 000 000 000 v[1]^2 v[2] / 2 999 999 994 000 000 003 - 1/3 v[1]^2 v[3] + v[1] v[2] v[4] - 1 000 000 000 v[8] / 2 999 999 994 000 000 003) // Expand

Out[•]= -v[1]^2 v[2] - 999 999 998 000 000 001 v[1]^2 v[3] / 1 000 000 000 + 2 999 999 994 000 000 003 v[1] v[2] v[4] / 1 000 000 000 - v[8]
```

In[•]:=
```
MyCollect[
 (-(2 (T^2 - 1))/(T ω[T])) (-v[1]^2 v[2] - ((T - 1)^2 v[1]^2 v[3])/T + (3 (T - 1)^2 v[1] v[2] v[4])/T - v[8]) /.
  With[{K = T}, v[i_] :>
    {ω[K], p1[K], (T ∂_T p1[K])/(T - T^-1), (T ∂_T ω[K])/(T - T^-1), T ∂_T (T ∂_T ω[K]), T ∂_T (T ∂_T (T ∂_T ω[K])), -2 c_{2,0}[K] +
      ω[K] c_{2,1}[K], ω[K] (c_{2,1}[K] - ω[K] c_{2,2}[K])/(-2 T + 2/T) (*,ω[K]^2 c_{2,2}[K]/2,OP2[K]*)}[[i]]],
 {c_{2,1}[T], c_{2,2}[T], p1[T], p1'[T]}, Simplify]
```

Out[•]=
$$-c_{2,1}[T]+\omega[T]\,c_{2,2}[T]+2\,(-1+T)^2\,\omega[T]\,p1'[T]-\frac{2\,(-1+T)\,p1[T]\,\left(-(1+T)\,\omega[T]+3\,(-1+T)\,T\,\omega'[T]\right)}{T}$$

In[•]:=
```
Simplify[{c_{2,1}[T], c_{2,2}[T]} /.
  Solve[ReleaseHold[Hold[(0 == c_{2,2}[K] (-3 c_{1,0}[K] ∂_T ω[K] + ω[K] ∂_T c_{1,0}[K]) +
        c_{2,1}[K] (-ω[K] ∂_T ω[K] + 2 T (∂_T ω[K])^2 - T ω[K] ∂_{T,T} ω[K])) ∧ (c_{2,1}[K] == ω[K] c_{2,2}[K] +
      2 (-1 + T)^2 ω[K] ∂_T p1[K] - (2 (-1 + T) p1[K] (-(1 + T) ω[K] + 3 (-1 + T) T ∂_T ω[K]))/T)] /.
    c_{1,0}[K] → (-(-1 + T)^2 p1[T] + T^2 ω[T] ω'[T])/T /. K → T],
  {c_{2,1}[T], c_{2,2}[T]}]]
```

Out[•]=
$$\left\{\left\{\frac{2\,(-1+T)\,p1[T]\,\left((1+T)\,\omega[T]-3\,(-1+T)\,T\,\omega'[T]\right)}{T}+2\,\omega[T]\,\left((-1+T)^2\,p1'[T]+2\,T^2\,\omega'[T]^2-T\,\omega[T]\,(\omega'[T]+T\,\omega''[T])\right),\right.\right.$$
$$\left.\left.-2\,T\,\left(-2\,T\,\omega'[T]^2+\omega[T]\,(\omega'[T]+T\,\omega''[T])\right)\right\}\right\}$$